Chapter 7

Design of Robust Compensators based on μ Theory

There are two main limitations in the use of $\mathcal{H}_\infty$ theory for compensator design. First, only full complex perturbations $\Delta(s) \in \mathbf{C}^{n \times m}$ can be treated in a non-conservative way in an $\mathcal{H}_\infty$ robust stability test. Second, robust performance can only be handled in a conservative way *even for full complex perturbations* since stability and performance can not be separated in the $\mathcal{H}_\infty$ structure. The conservatism depends on the uncertainty structure and on the condition number κ of the system. In this chapter, it will be demonstrated, that these limitations can be avoided by using the *structured singular value* μ.

First, the analysis problem will be considered, i.e., how given a compensator $K(s)$ robust stability and robust performance is verified using μ. Then, the synthesis problem will be discussed, i.e. how to find a compensator, which is optimal with respect to μ.

7.1 μ Analysis

7.1.1 Robust Stability

In the sequel, control problems that can be represented in the block diagram structure shown in Figure 7.1 will be considered. This structure will be refered to as the $N\Delta K$ *structure.*

The similarity between the $N\Delta K$ and the 2×2 block structure is obvious. Here, however, $\Delta(s)$ will not be restricted to be a full complex block. Instead, it is assumed that $\Delta(s)$ has a certain *block diagonal* structure. Indeed, assume that $\Delta(s)$ belongs to the following bounded subset:

$$\mathbf{B\Delta} = \{\Delta(s) \in \mathbf{\Delta} \,|\bar{\sigma}(\Delta(j\omega)) < 1\} \tag{7.1}$$

where $\mathbf{\Delta}$ is defined as:

$$\mathbf{\Delta} = \big\{\mathrm{diag}\left(\delta_1^r I_{r_1}, \cdots, \delta_{m_r}^r I_{r_{m_r}}, \delta_1^c I_{r_{m_r+1}}, \cdots, \delta_{m_c}^c I_{r_{m_r+m_c}}, \Delta_1, \cdots, \Delta_{m_C}\right) | \\ \delta_i^r \in \mathbf{R}, \delta_i^c \in \mathbf{C}, \Delta_i \in \mathbf{C}^{r_{m_r+m_c+i} \times r_{m_r+m_c+i}}\big\} \tag{7.2}$$

Thus, both real and complex perturbations which influences the nominal system via the $N\Delta K$ structure are considered. Very general robust stability problems can be formulated via this structure, e.g. parametric uncertainty, see Example 7.1 on the following page. Obviously, the

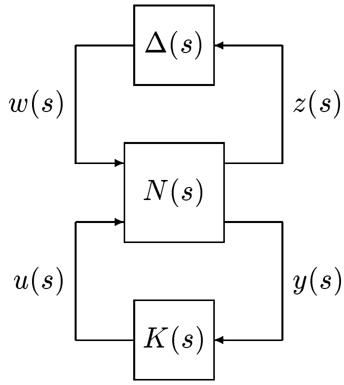

Figure 7.1: *$N\Delta K$ formulation of the robust stability problem.*

block diagonal structure of $\Delta(s)$ allow for a much more detailed uncertainty description, than if $\Delta(s)$ simply consists of one full complex block. Note, that a single full complex block of course is just a special case of the set $\mathbf{\Delta}$.

Example 7.1 (Diagonal perturbation formulation I)
This example is a slightly modified version of an example given in [Hol94]. Assume that the system $G(s)$ is given by:

$$G(s) = \frac{\alpha}{\beta s + 1} \tag{7.3}$$

where the DC gain α and the time constant β only are known with 10 % uncertainty:

$$\alpha = [27.0, 33.0], \qquad \beta = [0.9, 1.1] \tag{7.4}$$

Expressing α and β by their nominal values along with two perturbations Δ_α and Δ_β for which $|\Delta_{\alpha,\beta}| \leq 1$ can be obtained as:

$$\alpha = 30\left(1 + \frac{1}{10}\Delta_\alpha\right) \tag{7.5}$$

$$\beta = 1.0\left(1 + \frac{1}{10}\Delta_\beta\right) \tag{7.6}$$

where

$$\Delta_\alpha \in [-1, +1], \qquad \Delta_\beta \in [-1, +1] \tag{7.7}$$

Let $\mathbf{B\Delta}$ denote the set $[-1, +1]$. Then, the transfer function $G(s)$ can be written as:

$$G(s) = \frac{30\,(1 + 0.1\Delta_\alpha)}{(1 + 0.1\Delta_\beta)\,s + 1} \tag{7.8}$$

with:

$$\Delta_\alpha, \Delta_\beta \in \mathbf{B\Delta} \tag{7.9}$$

In block diagram form, $G(s)$ can be represented as shown in Figure 7.2 on the next page.

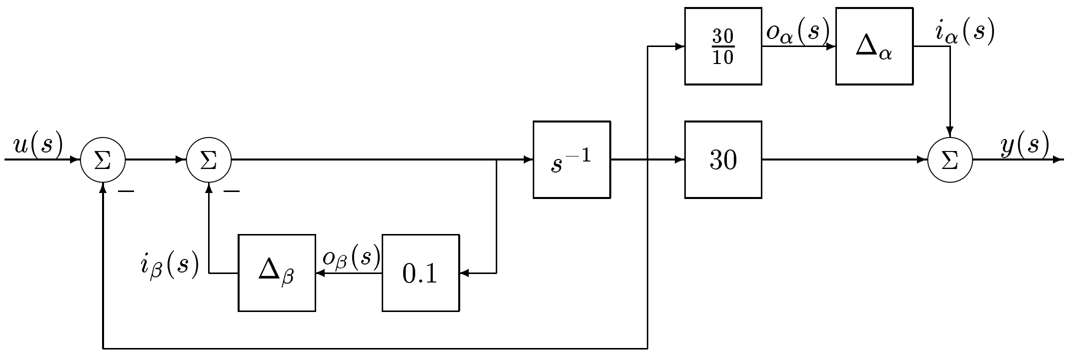

Figure 7.2: *Example 7.1: Block diagram representation of $G(s)$.*

To determine the $N\Delta K$ formulation, the Δ block in Figure 7.2 is removed, and the transfer functions from the three inputs $i_\alpha(s)$, $i_\beta(s)$, and $u(s)$ to the three outputs $o_\alpha(s)$, $o_\beta(s)$, and $y(s)$ are determined. Standard block diagram manipulation in matrix form gives:

$$\begin{bmatrix} o_\alpha(s) \\ o_\beta(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{30}{10(s+1)} & \frac{30}{10(s+1)} \\ 0 & -\frac{s}{10(s+1)} & \frac{s}{10(s+1)} \\ 1 & -\frac{30}{s+1} & \frac{30}{s+1} \end{bmatrix} \begin{bmatrix} i_\alpha(s) \\ i_\beta(s) \\ u(s) \end{bmatrix} \tag{7.10}$$

The uncertainty blocks are given as:

$$\begin{bmatrix} i_\alpha(s) \\ i_\beta(s) \end{bmatrix} = \begin{bmatrix} \Delta_\alpha o_\alpha(s) \\ \Delta_\beta o_\beta(s) \end{bmatrix} = \begin{bmatrix} \Delta_\alpha & 0 \\ 0 & \Delta_\beta \end{bmatrix} \begin{bmatrix} o_\alpha(s) \\ o_\beta(s) \end{bmatrix} \tag{7.11}$$

Now, let $w(s)$, $z(s)$, $N(s)$, and $\Delta(s)$ be given by:

$$w(s) = \begin{bmatrix} i_\alpha(s) \\ i_\beta(s) \end{bmatrix} \tag{7.12}$$

$$z(s) = \begin{bmatrix} o_\alpha(s) \\ o_\beta(s) \end{bmatrix} \tag{7.13}$$

$$N(s) = \begin{bmatrix} 0 & -\frac{30}{10(s+1)} & \frac{30}{10(s+1)} \\ 0 & -\frac{s}{10(s+1)} & \frac{s}{10(s+1)} \\ 1 & -\frac{30}{s+1} & \frac{30}{s+1} \end{bmatrix} \tag{7.14}$$

$$\Delta(s) = diag\{\Delta_\alpha, \Delta_\beta\} = \begin{bmatrix} \Delta_\alpha & 0 \\ 0 & \Delta_\beta \end{bmatrix} \tag{7.15}$$

The perturbed system can now be described as:

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = N(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} \tag{7.16}$$

$$w(s) = \Delta(s)z(s) \tag{7.17}$$

and can immediately be put into the $N\Delta K$ structure.

Example 7.2 (Diagonal perturbation formulation II)
Now, consider a standard second order system:

$$G(s) = \frac{\alpha\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} \tag{7.18}$$

Assume that the gain α, the damping ζ, and the resonance frequency ω_n are not known exactly, but only such that:

$$\alpha = \alpha_o(1 + \delta_\alpha \Delta_\alpha) \tag{7.19}$$
$$\zeta = \zeta_o(1 + \delta_\zeta \Delta_\zeta) \tag{7.20}$$
$$\omega_n = \omega_{n_o}(1 + \delta_\omega \Delta_\omega) \tag{7.21}$$

where

$$\Delta_\alpha, \Delta_\zeta, \Delta_\omega \in \mathbf{B\Delta} \qquad \mathbf{B\Delta} = [-1; +1] \tag{7.22}$$

Hence, α_o, ζ_o, and ω_{n_o} are the nominal values, whereas δ_α, δ_ζ, and δ_ω are the relative uncertainties.

A representation of $G(s)$ in transfer function form is given in Figure 7.3 on the following page. Due to the increased complexity relative to the first order example in Example 7.1 on page 78, it is more convenient to work with state space representations. Define the states as:

$$x_1 = \dot{y}, \qquad x_2 = y \tag{7.23}$$

It can then be shown that a state space representation for $G(s)$ is given by:

$$N(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$
$$= \left[\begin{array}{cc|cccccc}
-2\zeta_o\omega_{n_o} & -\omega_{n_o}^2 & \omega_{n_o}^2 & \omega_{n_o} & 1 & 2\zeta_o\omega_{n_o}^2 & -\omega_{n_o}^2 & \alpha\omega_{n_o}^2 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha\delta_\alpha \\
-2\delta_\omega\zeta_o & -\delta_\omega\omega_{n_o} & \delta_\omega\omega_{n_o} & 0 & 0 & 2\delta_\omega\zeta_o\omega_{n_o} & -\delta_\omega\omega_{n_o} & \delta_\omega\alpha\omega_{n_o} \\
-2\delta_\omega\zeta_o\omega_{n_o} & -\delta_\omega\omega_{n_o}^2 & \delta_\omega\omega_{n_o}^2 & \delta_\omega\omega_{n_o} & 0 & 2\delta_\omega\zeta_o\omega_{n_o}^2 & -\delta_\omega\omega_{n_o}^2 & \delta_\omega\alpha\omega_{n_o}^2 \\
\delta_\omega\omega_{n_o}^{-1} & 0 & 0 & 0 & 0 & \delta_\omega & 0 & 0 \\
2\delta_\zeta\zeta_o\omega_{n_o}^{-1} & 0 & 0 & 0 & 0 & -2\delta_\zeta\zeta_o & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \tag{7.24}$$

Now, proceeding as in Example 7.1 on page 78, define:

$$w(s) = \left[\begin{array}{ccccc} i_1(s) & i_2(s) & i_3(s) & i_4(s) & i_5(s) \end{array}\right]^T \tag{7.25}$$
$$z(s) = \left[\begin{array}{ccccc} o_1(s) & o_2(s) & o_3(s) & o_4(s) & o_5(s) \end{array}\right]^T \tag{7.26}$$
$$\Delta(s) = \text{diag}\ \{\Delta_\alpha, \Delta_\omega, \Delta_\omega, \Delta_\omega, \Delta_\zeta\} \tag{7.27}$$

Then the perturbed second order system is given by:

$$\left[\begin{array}{c} z(s) \\ y(s) \end{array}\right] = N(s) \left[\begin{array}{c} w(s) \\ u(s) \end{array}\right] \tag{7.28}$$
$$w(s) = \Delta(s)z(s) \tag{7.29}$$

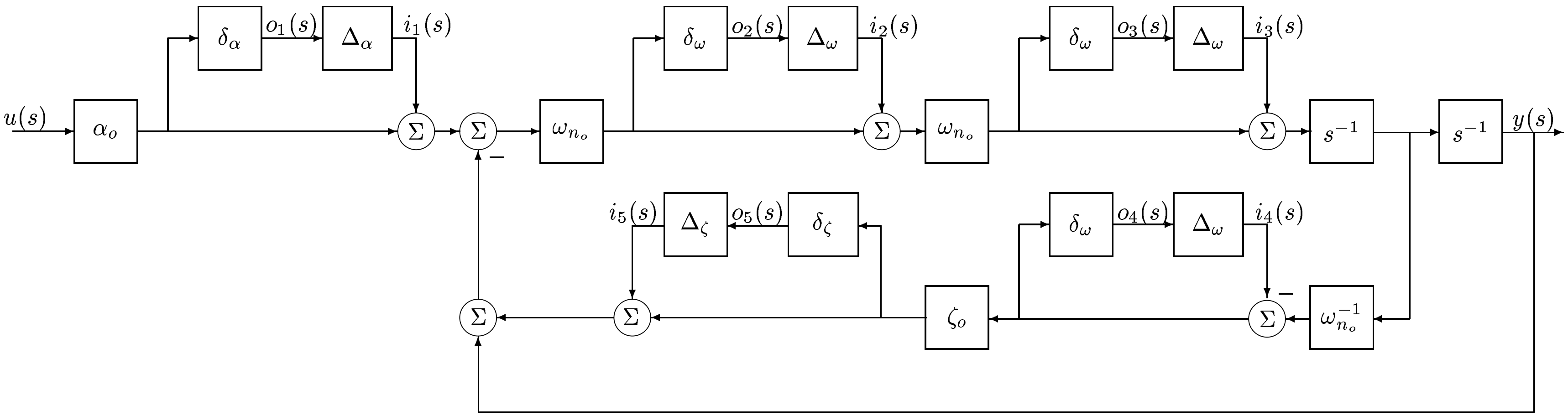

Figure 7.3: *Perturbed second order system in transfer function form.*

and can immediately be put into the $N\Delta K$ structure. Note, that in this case, the block structure for $\Delta(s)$ contains repeated scalar blocks.:

$$\boldsymbol{\Delta} = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 I_{3\times 3} & 0 \\ 0 & 0 & \delta_3 \end{bmatrix} \tag{7.30}$$

In general, for an arbitrary uncertain system, several equivalent $N\Delta K$ formulations will be possible, which can contain different $\Delta(s)$ structures. It might be difficult to determine a minimal *formulation, where the size of $\Delta(s)$ is the smallest possible.*

As illustrated by the above two examples, highly structured uncertainty models can be represented in the $N\Delta K$ structure. Unfortunately, extracting the uncertainty blocks can involve some tedious algebra. In the MATLAB$^{\text{TM}}$ μ toolbox there exists, however a very handy `m` function `sysic.m`, which facilitates an automization of this process.

Dynamical uncertainty can also be included via complex blocks of appropriate dimensions.

Now, let $F_l(N(s), K(s)) = P(s)$ denote the transfer function obtained by closing the lower loop in Figure 7.1 on page 78. $P(s)$ is the *generalized closed loop transfer function* and is given by:

$$P(s) = F_l(N(s), K(s)) \tag{7.31}$$

$$= N_{11}(s) + N_{12}(s)K(s)\left(I - N_{22}(s)K(s)\right)^{-1} N_{21}(s) \tag{7.32}$$

Then, given a structured uncertainty $\Delta(s) \in \mathbf{B\Delta}$, robust stability is determined through the following theorem, which is a generalization of the $\mathcal{H}_\infty$ robust stability theorem (see Theorem 5.2 on page 52).

Theorem 7.1 *Assume that the system $P(s)$ is stable, and that the perturbation $\Delta(s)$ is of such nature, that the closed loop system is stable if and only if the Nyquist curve for* $\det(I - P(s)\Delta(s))$ *does not encircle the origin. Then the closed loop system in Figure 7.1 on page 78 is stable for all perturbations $\Delta(s) \in \mathbf{B\Delta}$ if and only if*

$$\det(I - P(j\omega)\Delta(j\omega)) \neq 0 \qquad \forall\omega,\ \forall\Delta(j\omega) \in \mathbf{B\Delta} \tag{7.33}$$

$$\Leftrightarrow \quad \rho(P(j\omega)\Delta(j\omega)) < 1 \qquad \forall\omega,\ \forall\Delta(j\omega) \in \mathbf{B\Delta} \tag{7.34}$$

$$\Leftarrow \quad \bar{\sigma}(P(j\omega)) < 1 \qquad \forall\omega \tag{7.35}$$

***Proof of Theorem* 7.1** The proof follows immediately from the proof for the $\mathcal{H}_\infty$ robust stability theorem (Theorem 5.2 on page 52) with $\Delta(s) \in \mathbf{B\Delta}$. □

Note, that (7.35) is only a sufficient condition for robust stability. Necessity of the corresponding condition for unstructured uncertainties follows from the fact, at the unstructured set contains *all* $\Delta(s)$ with $\bar{\sigma}(\Delta(j\omega)) \leq 1$. Now, however, the perturbation set is restricted to $\Delta(s) \in \mathbf{B\Delta}$ and, thus, the condition (7.35) might in general be arbitrarily conservative. Rather than a robust stability condition based on singular values, a condition is required which takes the structure of the perturbation into consideration. This is precisely the virtue of the structured singular value μ.

Given any matrix $P \in \mathbf{C}^{n\times m}$ the positive real function μ is defined by:

$$\mu_{\boldsymbol{\Delta}}(P) \triangleq \frac{1}{\min\left\{\bar{\sigma}(\Delta) : \Delta \in \boldsymbol{\Delta}, \det(I - P\Delta) = 0\right\}} \tag{7.36}$$

except if no $\Delta \in \mathbf{\Delta}$ makes $I - P\Delta$ singular ($\det(I - P\Delta) = 0$); in this case, by definition $\mu_{\mathbf{\Delta}}(P) = 0$. Hence, $1/\mu_{\mathbf{\Delta}}(P)$ is the 'magnitude' of the smallest perturbation Δ measured by its singular value $\bar{\sigma}(\Delta)$ making $I - P\Delta$ singular. If $P(s)$ is a transfer matrix, $1/\mu_{\mathbf{\Delta}}(P(j\omega))$ can be interpreted as the magnitude of the smallest perturbation which moves the characteristic loci of $P(s)$ into the Nyquist point $(-1, 0)$ at the angular frequency ω.

From the definition of μ and Theorem 7.1 on the page before the following theorem for determining robust stability can be formulated (see also [DP87, PD93]:

Theorem 7.2 (Robust stability with μ) *Assume that the system $P(s)$ is stable, and that the perturbation $\Delta(s)$ is such that the closed loop system is stable if and only if the Nyquist curve for $\det(I - P(s)\Delta(s))$ does not encircle the origin. Then, the closed loop system in Figure 7.1 on page 78 is stable for all perturbations $\Delta(s) \in \mathbf{B\Delta}$ if and only if*

$$\|\mu_{\mathbf{\Delta}}(P(s))\|_{\infty} \le 1 \tag{7.37}$$

where:

$$\|\mu_{\mathbf{\Delta}}(P(s))\|_{\infty} \stackrel{\triangle}{=} \sup_{\omega} \mu_{\mathbf{\Delta}}(P(j\omega)) \tag{7.38}$$

7.1.2 Robust Performance

In order to analyze a system with respect to robust performance, the normalized exogenous disturbances $d'(s)$ and the normalized error signals $e'(s)$ are included in the $N\Delta K$ formulation. Now, a general framework for the analysis and synthesis of linear systems can be formulated, see Figure 7.4. Any linear combination of control inputs u, measured outputs y, disturbances d', error signals e', perturbations w and compensator K can be described via this 'generic' system.

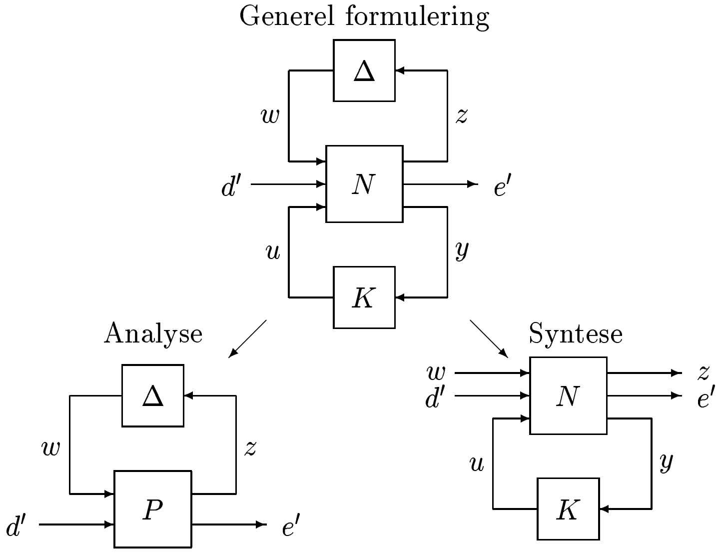

Figure 7.4: *A general framework for the analysis and for compensator design for linear systems.*

Within this framwork, analysis and design can be seen as two special cases, see Figure 7.4 on the preceding page. Like in the 2×2 block problem, scalings and weightings have been arbsorbed into the transfer matrix $N(s)$ such that $d'(s)$, $e'(s)$ and $\Delta(s)$ are normalized to one in norm. Note, that if $P(s)$ is partitioned into four blocks, consistent with the dimension of the two inputs w and d' and of the two outputs z and e', P_{11} can be identified as the transfer matrix $P(s)$ in Theorem 7.2 on the page before.

In the analysis of robust performance, the transfer matrix from $d'(s)$ to $e'(s)$ is studied. This transfer matrix is given by:

$$e'(s) = F_u(P(s), \Delta(s))d'(s) \tag{7.39}$$

$$= \left[P_{22}(s) + P_{21}(s)\Delta(s)\left(I - P_{11}(s)\Delta(s)\right)^{-1} P_{12}(s)\right] d'(s) \tag{7.40}$$

In (7.40), $P_{22}(s)$ is the weighted nominal performance transfer matrix (for example the output sensitivity) and, hence, $F_u(P(s), \Delta(s))$ is the weighted perturbed performance transfer matrix. The robust performance measure can now be formulated using (7.40) as:

$$\|F_u(P(s), \Delta(s))\|_{\mathcal{H}_\infty} = \sup_{\omega} \bar{\sigma}\left(F_u(P(j\omega), \Delta(j\omega))\right) < 1 \quad \forall \Delta(j\omega) \in \mathbf{B\Delta} \tag{7.41}$$

Note, that the condition for robust performance is formulated as a singular value bound, just as a robust stability condition for unstructured uncertainty. Thus, it can be concluded that *the robust performance condition* (7.41) *is satisfied if and only if the system $F_u(P(s), \Delta(s))$ is robustly stable in the face of a norm bounded perturbation $\Delta_p(s)$ with $\bar{\sigma}(\Delta_p(j\omega)) \leq 1$,* $\forall \omega$. Hence, by augmenting the perturbation structure by a full complex 'performance block' $\Delta_p(s)$, robust performance can be verified via a robust stability condition. Furthermore, this augmentation of the uncertainty structure can be carried out in a quite natural way using μ, as the admissible structure for μ is precisely a block diagonal one.

This facilitates the following theorem for assessing robust performance, see also [DP87, PD93]:

Theorem 7.3 (Robust performance with $\boldsymbol{\mu}$) *Assume that performance specifications have been given as an $\mathcal{H}_\infty$ specification of the transfer matrix from $d'(s)$ to $e'(s)$ (typically a weighted sensitivity specification) of the form:*

$$\|F_u(P(s), \Delta(s))\|_{\mathcal{H}_\infty} = \sup_{\omega} \bar{\sigma}\left(F_u(P(j\omega), \Delta(j\omega))\right) < 1 \tag{7.42}$$

Then the perturbed closed loop system $F_u(P(s), \Delta(s))$ and the performance specification $\|F_u(P(s), \Delta(s))\|_{\mathcal{H}_\infty} < 1$, $\forall \Delta(s) \in \mathbf{B\Delta}$ if and only if:

$$\left\|\mu_{\tilde{\mathbf{\Delta}}}(P(s))\right\|_\infty \leq 1 \tag{7.43}$$

where the perturbation structure has been augmented by a full complex performance block:

$$\tilde{\mathbf{\Delta}} = \left\{\mathrm{diag}\left(\Delta, \Delta_p\right) \middle| \Delta \in \mathbf{\Delta}, \Delta_p \in \mathbf{C}^{k \times k}\right\} \tag{7.44}$$

Theorem 7.3 is one of the reasons why the $\mathcal{H}_\infty$ norm is popular as a measure of performance. Indeed, if the uncertainty is bounded by the largest singular value, it is possible via μ to check for robust stability *and robust performance* in a non-conservative way. If the uncertainty is modeled in a precise way, i.e. if all systems $G_\Delta(s) \in \mathcal{G}$ could appear in practice, then the μ condition for robust performance is both necessary and sufficient. Thus, the μ theorems provide less conservative conditions for robust performance compared to the corresponding $\mathcal{H}_\infty$ conditions. Stability and performance can be separated, far more detailed uncertainty descriptions can be formulated due to the block diagonal structure of $\mathbf{\Delta}$, and non-conservative conditions for robust performance are obtained, even for ill-conditioned systems ($\kappa(G(j\omega)) \gg 1$).

(7.43) on the page before provides a simple test for robust performance. If $\mu_{\tilde{\mathbf{\Delta}}}(P(j\omega))$ is plotted against frequency, it is easy to check whether the condition (7.43) in Theorem 7.3 on the preceding page is satisfied.

Since $\Delta_1 = \text{diag}\{\Delta, 0\}$ and $\Delta_2 = \text{diag}\{0, \Delta_p\}$ are special cases of the general structure $\Delta \in \tilde{\mathbf{\Delta}}$ it is obvious that:

$$\mu_{\tilde{\mathbf{\Delta}}}(P(j\omega)) \geq \max\{\mu_{\mathbf{\Delta}}(P_{11}(j\omega)), \mu_{\mathbf{\Delta}_p}(P_{22}(j\omega)) = \bar{\sigma}(P_{22}(j\omega))\} \tag{7.45}$$

which means that a necessary condition for robust performance is that the closed loop system must be robustly stable, and that the nominal system satisfies the performance specifications.

7.1.3 Computing μ

As illustrated above, μ is a very useful tool for determining robust stability and robust performance in the face of structured as well as unstructed uncertainties. Unfortunately, the computation of μ itself is a complicated problem, which does not allow a general mathematical solution. The trouble is, that (7.36) on page 82 can not be used directly for computing μ since the optimization problem involved in general will have several local optima [DP87, FTD91].

Upper and lower bounds for μ, however, can be computed both for purely complex perturbation sets ($m_r = 0$ in (7.2) on page 77) and for mixed real and complex perturbation sets. Algorithms for computing these bounds were the subject of intense research activities in the beginning of the 1990's, see e.g. [DP87, YND91]. In the sequel some of the bounds will be presented. To avoid making the notation any more complicated than required, it will be assumed that the generalized system $P(s)$ is square, $P(s) \in \mathbf{C}^{n\times n}$.

7.1.3.1 μ for complex perturbations

First, the computation of μ will be considered in the case, where the perturbation structure consists entirely of complex blocks, i.e. when $m_r = 0$ in (7.2) on page 77. It is not difficult to show that $\mu_{\mathbf{\Delta}}(P)$ can be computed by standard functions, when $\mathbf{\Delta}$ is one of the following two sets, see e.g. [ZDG96]:

- If $\mathbf{\Delta} = \{\delta^c I_n \,|\, \delta^c \in \mathbf{C}\}$ ($m_r = 0$, $m_c = 1$, $m_C = 0$ in (7.2)), then $\mu_{\mathbf{\Delta}}(P) = \rho(P)$, the spectral radius of P (the largest absolute value of any eigenvalue of P, $\rho(P) = \max_i |\lambda_i(P)|$).
- If $\mathbf{\Delta} = \{\Delta \,|\, \Delta \in \mathbf{C}^{n\times n}\}$ ($m_r = 0$, $m_c = 0$, $m_C = 1$ in (7.2)), then $\mu_{\mathbf{\Delta}}(P) = \bar{\sigma}(P)$, the largest singular value of P.

For a general complex perturbation $\mathbf{\Delta}$ the following holds:

$$\{\delta^c I_n \,|\, \delta^c \in \mathbf{C}\} \subset \mathbf{\Delta} \subset \{\Delta \,|\, \Delta \in \mathbf{C}^{n\times n}\} \tag{7.46}$$

Therefore:

$$\rho(P) \leq \mu_{\mathbf{\Delta}}(P) \leq \bar{\sigma}(P) \tag{7.47}$$

These bounds, however, are yet not satisfactory, as the discrepancy between $\rho(P)$ and $\bar{\sigma}(P)$ can be arbitrarily large. Thus, the bounds of (7.47) have to be refined. This can be done through certain transformations of P which *do not affect* $\mu_{\mathbf{\Delta}}(P)$ but, nevertheless, modifies $\rho(P)$ and $\bar{\sigma}(P)$. To that end, define the following subsets of $\mathbf{C}^{n \times n}$:

$$\mathbf{Q} = \left\{ Q \in \mathbf{\Delta} \,\middle|\, m_r = 0, \delta_i^{c^*} \delta_i^c = 1, \Delta_i^* \Delta_i = I_{r_{m_c+i}} \right\} \tag{7.48}$$

$$\mathbf{D} = \Big\{ \operatorname{diag}\left(D_1, \cdots, D_{m_c}, d_1 I_{r_{m_c+1}}, \cdots, d_{m_C} I_{r_{m_c+m_C}} \right) \\ \Big| D_i \in \mathbf{C}^{r_i \times r_i}, D_i^* = D_i > 0, d_i \in \mathbf{R}, d_i > 0 \Big\} \tag{7.49}$$

It can now be shown (see e.g. the original paper on μ by Doyle [Doy82]) that for any $\Delta \in \mathbf{\Delta}$ (for which $m_r = 0$), $Q \in \mathbf{Q}$ and $D \in \mathbf{D}$ the following holds:

$$Q^* \in \mathbf{Q}, \quad Q\Delta \in \mathbf{\Delta}, \quad \Delta Q \in \mathbf{\Delta}, \quad \bar{\sigma}(Q\Delta) = \bar{\sigma}(\Delta Q) = \bar{\sigma}(\Delta), \tag{7.50}$$

$$D\Delta = \Delta D \tag{7.51}$$

From (7.50) and (7.51), the following theorem can be derived.

Theorem 7.4 (Upper and lower bounds for μ) *For any $Q \in \mathbf{Q}$ and $D \in \mathbf{D}$ the following holds:*

$$\mu_{\mathbf{\Delta}}(PQ) = \mu_{\mathbf{\Delta}}(QP) = \mu_{\mathbf{\Delta}}(P) = \mu_{\mathbf{\Delta}}(DPD^{-1}) \tag{7.52}$$

Thus, the bounds in (7.47) can be refined as:

$$\max_{Q \in \mathbf{Q}} \rho(QP) \leq \mu_{\mathbf{\Delta}}(P) \leq \inf_{D \in \mathbf{D}} \bar{\sigma}(DPD^{-1}) \tag{7.53}$$

The lower bound $\max_{Q \in \mathbf{Q}} \rho(QP)$ is in fact an identity ($\max_{Q \in \mathbf{Q}} \rho(QP) = \mu_{\mathbf{\Delta}}(P)$), but unfortunately, the function $\rho(QP)$ is not convex, and in general it will have several local maxima. Hence, a numerical search algorithm is not guaranteed to find μ but rather just a lower bound. On the other hand, the upper bound is a convex problem, and thus, the global minimum $\inf_{D \in \mathbf{D}} \bar{\sigma}(DPD^{-1})$ can in principle always be determined. Unfortunately, the upper bound is sometimes strict, i.e. the global infimum might not be equal to μ. It can be shown, that for especially simple perturbation structure, i.e. for $m_r = 0$ and $2m_c + m_C \leq 3$, the upper bound always equals μ.

However, for struktures with $2m_c + m_C > 3$, and for most matrices P, μ will be strictly less than $\inf_{D \in \mathbf{D}} \bar{\sigma}(DPD^{-1})$. On the other hand, numerical experience indicate that even for $2m_c + m_C > 3$ the upper bound is usually not highly conservative.

With the MATLAB™ μ Analysis and Synthesis Toolbox [BDG+93], commercial software is now available for computing the bounds of Theorem 7.4. For practical compensator design (at least for purely complex perturbations), the mathematical problems involved in the computation of μ seems to be of less significance.

7.1.3.2 μ with mixed perturbations

The solution to the mixed[1] μ problem has also been the subject of an intense research effort during the past ten years, see e.g. [FTD91, YND91, YND92, You93]. In these lecture notes, the computation of the bounds for the mixed μ case will not be presented in full details (one reference is [You93]), just a few of the more important results will be stated. To that end, define the following sets:

$$\mathbf{Q} = \left\{ Q \in \mathbf{\Delta} \,\middle|\, \delta_i^r \in [-1;1], \delta_i^{c^*}\delta_i^c = 1, \Delta_i^*\Delta_i = I_{r_{m_r+m_c+i}} \right\} \tag{7.54}$$

$$\begin{aligned}\mathbf{D} = \big\{ &\mathrm{diag}\left(D_1, \cdots, D_{m_r+m_c}, d_1 I_{r_{m_r+m_c+1}}, \cdots, d_{m_C} I_{r_m}\right) \\ &\big| D_i \in \mathbf{C}^{r_i\times r_i}, D_i^* = D_i > 0, d_i \in \mathbf{R}, d_i > 0 \big\}\end{aligned} \tag{7.55}$$

$$\mathbf{G} = \left\{ \mathrm{diag}\left(G_1, \cdots, G_{m_r}, O_{r_{m_r+1}}, \cdots, O_{r_m}\right) \big| G_i \in \mathbf{C}^{r_i\times r_i}, G_i = G_i^* \right\} \tag{7.56}$$

$$\begin{aligned}\hat{\mathbf{D}} = \big\{ &\mathrm{diag}\left(D_1, \cdots, D_{m_r+m_c}, d_1 I_{r_{m_r+m_c+1}}, \cdots, d_{m_C} I_{r_m}\right) \\ &\big| D_i \in \mathbf{C}^{r_i\times r_i}, \det(D_i) \neq 0, d_i \in \mathbf{C}, d_i \neq 0 \big\}\end{aligned} \tag{7.57}$$

$$\hat{\mathbf{G}} = \{ \mathrm{diag}\left(g_1, \cdots, g_{n_r}, O_{n_c}\right) | g_i \in \mathbf{R} \} \tag{7.58}$$

where $r_m = r_{m_r+m_c+m_C}$, $n_r = \sum_{i=1}^{m_r} r_i$ and $n_c = n - n_r$. Note, that for consistency with $P(s)$, $\sum_{i=1}^{m} r_i = n$ is required.

Then, the following upper and lower bounds for mixed μ apply:

Theorem 7.5 (Upper and lower bounds for mixed μ [FTD91]) *Let $\bar{\lambda}_R$ be the largest real eigenvalue of P and let $\rho_R(P)$ denote the spectral radius of P:*

$$\rho_R(P) \stackrel{\triangle}{=} \max\{|\lambda_R(P)| \;:\; \lambda_R(P) \textit{ is a real eigenvalue of } P\} \tag{7.59}$$

If P does not have any real eigenvalues, then $\rho_R(P) = 0$. Assume further that α_* is the result of the following minimization problem:

$$\alpha_* = \inf_{D\in\mathbf{D}, G\in\mathbf{G}} \min_{\alpha\in\mathbf{R}} \left\{ \alpha \,\middle|\, \bar{\lambda}_R\left(P^*DP + j\left(GP - P^*G\right) - \alpha D\right) \leq 0 \right\} \tag{7.60}$$

Then:

$$\rho_R(P) \leq \mu_{\mathbf{\Delta}}(P) \leq \sqrt{\max(0, \alpha_*)} \tag{7.61}$$

Note, that the computation of the upper bound (7.60) involves a Linear Matrix Inequality (LMI). A number of numerical methods exists to tackle such minimizations. These require, however, even for relatively modest problems ($n \leq 100$), an optimization over scalings and $G(s)$, which can contain several thousands of parameters. Hence, dealing with such problems within reasonable computational times, require that the structure of the mixed μ problem is exploited to a wider extent, see e.g. [YND92]. Various reformulations of the upper bound problem are given in Theorem 7.6 on the next page.

[1]The μ synthesis problem in the presence of perturbations with both real and complex blocks, is often refered to simply as the mixed μ problem.

Theorem 7.6 (Reformulating the mixed μ upper bound) *Assume that a matrix $P \in \mathbf{C}^{n\times n}$ and a real positive scalar $\beta > 0$ are given. Further for any $D \in \mathbf{C}^{n\times n}$, let $P_D = DPD^{-1}$. Then the following statements are equivalent:*

1. *There exist matrices $D_1 \in \mathbf{D}$ and $G_1 \in \mathbf{G}$ such that:*

$$\bar{\lambda}_R\left(P^* D_1 P + j(G_1 P - P^* G_1) - \beta^2 D_1\right) \leq 0 \tag{7.62}$$

2. *There exist matrices $D_2 \in \mathbf{D}$ and $G_2 \in \mathbf{G}$ (or $D_2 \in \hat{\mathbf{D}}$ and $G_2 \in \hat{\mathbf{G}}$) such that:*

$$\bar{\lambda}_R\left(P_{D_2}^* P_{D_2} + j(G_2 P_{D_2} - P_{D_2}^* G_2)\right) \leq \beta^2 \tag{7.63}$$

3. *There exist matrices $D_3 \in \mathbf{D}$ and $G_3 \in \mathbf{G}$ (or $D_3 \in \hat{\mathbf{D}}$ and $G_3 \in \hat{\mathbf{G}}$) such that:*

$$\bar{\sigma}\left[\left(\frac{P_{D_3}}{\beta} - jG_3\right)\left(I + G_3^2\right)^{-\frac{1}{2}}\right] \leq 1 \tag{7.64}$$

4. *There exist matrices $D_4 \in \mathbf{D}$ and $G_4 \in \mathbf{G}$ (or $D_4 \in \hat{\mathbf{D}}$ and $G_4 \in \hat{\mathbf{G}}$) such that:*

$$\bar{\sigma}\left[\left(I + G_4^2\right)^{-\frac{1}{4}}\left(\frac{P_{D_4}}{\beta} - jG_4\right)\left(I + G_4^2\right)^{-\frac{1}{4}}\right] \leq 1 \tag{7.65}$$

A proof for Theorem 7.6 can be found in [You93]. Using Theorem 7.6, alternative formulations can easily be found for the mixed μ upper bound. For example, the upper bound, which is implemented in the MATLAB™ μ toolbox is derived from (7.65). Define β^* as:

$$\beta^* = \inf_{\beta \in \mathbf{R}_+, G \in \hat{\mathbf{G}}, D \in \hat{\mathbf{D}}} \left\{\beta \,|\, \bar{\sigma}\left(P_{DG}\right) \leq 1\right\} \tag{7.66}$$

where P_{DG} is given as:

$$P_{DG} = \left(I + G^2\right)^{-\frac{1}{4}}\left(\frac{DPD^{-1}}{\beta} - jG\right)\left(I + G^2\right)^{-\frac{1}{4}} \tag{7.67}$$

Then:

$$\max_{Q \in \mathbf{Q}} \rho(QP) \leq \mu_{\mathbf{\Delta}}(P) \leq \beta^* \tag{7.68}$$

7.2 μ synthesis

For compensator synthesis, it is convenient to partition the transfer matrix $F_l(N, K)$ from $[w, d']^T$ to $[z, e']^T$ as:

$$\begin{bmatrix} z(s) \\ e'(s) \end{bmatrix} = F_l(N(s), K(s)) \begin{bmatrix} w(s) \\ d'(s) \end{bmatrix} = \left[N_{11}(s) + N_{12}(s)K(s)\left(I - N_{22}(s)K(s)\right)^{-1} N_{21}(s)\right] \begin{bmatrix} w(s) \\ d'(s) \end{bmatrix} \tag{7.69}$$

Note, that $F_l(N(s), K(s)) = P(s)$. Applying Theorem 7.3 on page 84, it is seen that a nominally stabilizing compensator $K(s)$ achieves robust performance, if and only if the structured singular value μ for every frequency $\omega \in [0, \infty]$ satisfies:

$$\mu_{\hat{\mathbf{\Delta}}} \left(F_l(N(j\omega), K(j\omega))\right) < 1 \tag{7.70}$$

Thus, the optimal robust performance problem can be formulated as:

$$K(s) = \arg \min_{K(s)\in\mathcal{K}_S} \left\| \mu_{\hat{\mathbf{\Delta}}} \left(F_l(N(s), K(s))\right) \right\|_\infty \tag{7.71}$$

where $\mathcal{K}_S$ is the set of all nominally stabilizing compensators.

7.2.1 Complex μ Synthesis – D-K iteration

Unfortunately, the optimization (7.71) can not be directly evaluated for the simple reason, that μ can not be computed exactly. Instead, an upper bound problem can be formulated as:

$$K(s) = \arg \min_{K(j\omega)\in\mathcal{K}_S} \sup_{\omega} \inf_{D(\omega)\in\mathbf{D}} \left\{ \bar{\sigma}\left(D(\omega) F_l\left(N(j\omega), K(j\omega)\right) D^{-1}(\omega) \right) \right\} \tag{7.72}$$

Unfortunately, no solution has yet been found to the minimization problem (7.72). A practical approach to is the following iterative procedure. To determine $D(\omega)$ at a number of frequencies for a given compensator $K(s)$ is equivalent to solving the complex μ upper bound problem, which has a known solution. When these scalings have been found, a stable transfer matrix $D(s)$ can be fitted, such that $D(j\omega)$ is an approximation of $D(\omega)$ for all frequencies ω. It can even be assured that $D(s)$ is minimum phase, such that D^{-1} is also stable, as the phase of $D(s)$ is absorbed into the complex perturbations. In other words: it is only necessary to fit the amplitude of $D(j\omega)$.

For given matrix scalings $D(s)$, the problem is to find a compensator $K(s)$, which minimizes the norm $\|F_l(D(s)N(s)D^{-1}(s), K(s))\|_{\mathcal{H}_\infty}$ which is a standard $\mathcal{H}_\infty$ problem, for which the solution has been given in Theorem 6.1 on page 67.

For this compensator, new D scalings can be found, and the procedure starts all over again. If this iteration (known as the D-K iteration) converges to a specific compensator, this compensator is a good candidate for a near optimal μ compensator. Even though both the computation of D scales and of the optimal $\mathcal{H}_\infty$ compensator are convex optimization problems, D-K iteration is not a *jointly convex* optimization in $D(s)$ and $K(s)$. Thus, convergence can not be guaranteed. However, numerical experience shows that D-K iteration works well in practice. The D-K procedure can be formulated as follows:

Procedure 7.1 (D-K iteration)

1. *Given an augmented system $N(s)$, let $i = 1$ and $D_i^\star(\omega) = I, \forall\omega$.*

2. *Fit a stable minimum phase transfer matrix $D_i(s)$ to the pointwise scalings $D_i^\star(\omega)$. Augment $D_i(s)$ with a identity matrix, such that $D_i(s)$ becomes compatible with $N(s)$. Construct the system $N_{D_i}(s) = D_i(s)N(s)D_i^{-1}(s)$.*

3. *Find the $\mathcal{H}_\infty$ optimal compensator $K_i(s)$:*

$$K_i(s) = \arg \min_{K(s)\in\mathcal{K}_S} \|F_l(N_{D_i}(s), K(s))\|_{\mathcal{H}_\infty} \tag{7.73}$$

4. *Find the new scalings* $D^{\star}_{i+1}(\omega)$ *as a solution to the complex* μ *upper bound problem:*

$$D^{\star}_{i+1}(\omega) = \arg \min_{D(\omega)\in\mathbf{D}} \left\{\bar{\sigma}\left(D(\omega)F_l(N(j\omega), K_i(j\omega))D^{-1}(\omega)\right)\right\} \tag{7.74}$$

for every frequency ω.

5. *Compare* $D^{\star}_{i+1}(\omega)$ *and* $D^{\star}_{i}(\omega)$. *Stop, if they are 'close' (in magnitude). Otherwise, let* $i = i+1$ *and repeat the iteration from Step 2.*

Note, that the $\mathcal{H}_\infty$ solution is used to find the compensator in Step 3. The K step in the D-K iteration can be illustrated as shown in Figure 7.5.

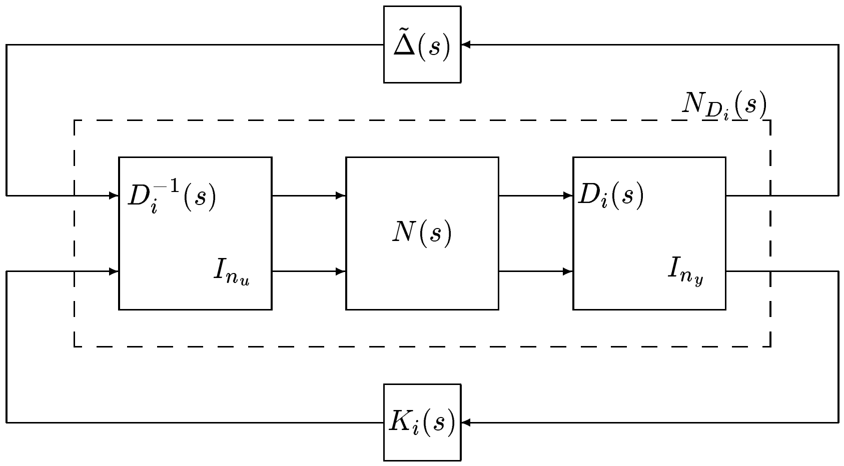

Figure 7.5: *K step in $D-K$* iteration. n_u and n_y are, respectively, the number of controllable inputs and the number of regulated outputs (error signals).

With the MATLAB™ *μ-Analysis and Synthesis Toolbox* commercially available software now exists which supports μ synthesis by D-K iteration. In early versions, only full complex blocks were supported. Note, that for repeated scalar blocks, the D scaling is a full matrix, and hence, the number of SISO transfer functions to be fitted grows rapidly.

Mixed μ synthesis is far more involved than the purely complex problem, and the present version of the μ toolbox does not support it fully. Recently, a couple of design methods for mixed μ synthesis has been proposed, see e.g. [You93, TC96].

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