Robust Control

Jakob Stoustrup

Section of Automation & Control
Department of Electronic Systems
Aalborg University

Outline

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1 Convexity

2 Linear Matrix Inequalities (LMIs)

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1. Convexity
2. Linear Matrix Inequalities (LMIs)
3. $\mathcal{H}_\infty$ and LMIs
4. Robust performance of interval systems

Convex sets

Definition

A set $\mathcal{S}$ is said to be *convex* if

$$\{x_1, x_2 \in \mathcal{S}\} \Rightarrow \{x := \alpha x_1 + (1-\alpha)x_2 \in \mathcal{S},\ \forall \alpha \in (0,1)\}$$

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Definition

Let $\mathcal{S}$ be a subset of a normed vector space and let $x_1, \ldots, x_n \in \mathcal{S}$. If $\alpha_1, \ldots, \alpha_n$ is a set of non-negative real numbers with $\sum_{i=1}^n \alpha_i = 1$, then

$$x := \sum_{i=1}^{n} \alpha_i x_i$$

is called a *convex combination* of $x_1, \ldots, x_n$.

Convex functions

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1. $\mathcal{S}$ is convex
2. For all $x_1, x_2 \in \mathcal{S}$ and $\alpha \in (0,1)$:

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$$

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Lemma

Assume that f is a convex function and that x_0 is a local minimum of f. Then x_0 is also a global minimum of f.
Even more importantly: such a local/global minimum can be found by simple Newton-like algorithms.

Outline

Linear Matrix Inequalities (LMIs)

A *Linear Matrix Inequality (LMI)* is an expression of the form:

$$F(x) := F_0 + x_1 F_1 + \ldots + x_m F_m > 0$$

where

- $x = (x_1, \ldots, x_m)$ is a vector of real numbers
- $F_0, \ldots, F_m$ are real symmetric matrices, i.e., $F_i = F_i^* \in \mathcal{R}^{n \times n}, i = 0, \ldots, m$
- $M > 0$ means that M is a positive definite matrix, i.e. $x^* M x > 0, \forall x \in \mathcal{R}^n, x \neq 0$

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Hence, an LMI takes the form

$$F(x) > 0$$

where F is an *affine* function that maps a vector into the set of symmetric matrices.

An LMI defines a *convex* subset of the vector space!

Systems of LMIs

Note that any finite set of LMIs can be written as one single LMI!

Indeed,

$$F_1(x) > 0, \ldots, F_k(x) > 0$$

can be written as

$$\begin{pmatrix} F_1(x) & 0 & \ldots & 0 \\ 0 & F_2(x) & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & F_k(x) \end{pmatrix} > 0$$

which is in itself a (giant) LMI.

Feasibility and optimization

Definition

An LMI

$$F(x) > 0$$

is said to be *feasible*, if there exists at least one x_0 such that $F(x_0) > 0$.

Consider $f : \mathcal{S} \rightarrow \mathcal{R}$ where $\mathcal{S} = \{x : F(x) > 0\}$. The problem of determining

$$V_{\text{opt}} = \inf_{x \in \mathcal{S}} f(x)$$

is known as an *optimization problem with an LMI constraint*.

LMI example: Lyapunov stability

Consider a linear system of the form:

$$\Sigma : \qquad \dot{x} = Ax$$

where $x(t) \in \mathcal{R}^n$ and $A \in \mathcal{R}^{n \times n}$. Then we have the following result:

Theorem (Lyapunov)

The system Σ is asymptotically stable if and only if there exists $X > 0$ such that

$$A^*X + XA < 0$$

This means that stability of Σ is equivalent to feasibility of the following LMI in X:

$$\begin{pmatrix} X & 0 \\ 0 & -A^*X - XA \end{pmatrix} > 0$$

LMI example: quadratic performance

Consider the linear autonomous system:

$$\dot{x} = Ax\,, \qquad x(0) = x_0$$

with a criterion function

$$J := \int_{t=0}^{\infty} x^*(t)Qx(t)\,dt$$

where $Q = Q^* \geq 0$, and A is Hurwitz. Then, it is possible to show that

$$J \leq x_0^* X x_0$$

for any solution $X = X^* > 0$ to the following LMI:

$$A^*X + XA + Q \leq 0$$

The smallest upper bound of J can be found by minimizing $x_0^* X x_0$ subject to the following LMI system:

$$\begin{cases} X > 0 \\ A^*X + XA + Q \leq 0 \end{cases}$$

Outline

$\mathcal{H}_\infty$ nominal performance

Proposition

Let $G(s)$ denotes the transfer function of the stable system

$$\Sigma : \quad \left\{ \begin{array}{ccccc} \dot{x} & = & Ax & + & Bw \\ z & = & Cx & + & Dw \end{array} \right.$$

The nominal $\mathcal{H}_\infty$ condition $\|G(\cdot)\|_\infty < \gamma$ is satisfied, if and only if there exists $X = X^* > 0$ such that

$$A^*X + XA + C^*C - (XB + C^*D)(D^*D - \gamma^2 I)^{-1}(B^*X + D^*C) < 0$$

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or, equivalently, if and only if the following LMI system is feasible:

$$X = X^* > 0$$

$$\begin{pmatrix} A^*X + XA + C^*C & XB + C^*D \\ B^*X + D^*C & D^*D - \gamma^2 I \end{pmatrix} < 0$$

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Strongly robust $\mathcal{H}_\infty$ performance

Definition

The time-varying uncertain dynamical system described by

$$\begin{array}{rcrcl} \dot{x} &=& A_\Delta x &+& B_\Delta w \\ z &=& C_\Delta x &+& D_\Delta w \end{array}$$

is said to satisfy the *strongly robust $\mathcal{H}_\infty$ performance criterion* if $\|D_\Delta\| < 1 \; \forall \Delta \in \boldsymbol{\Delta}$ and there exists a constant symmetric matrix $X > 0$ such that

$$\begin{aligned} & A_\Delta^* X + X A_\Delta + C_\Delta^* C_\Delta \\ & \qquad\qquad + (X B_\Delta + C_\Delta^* D_\Delta) R_\Delta^{-1} (B_\Delta^* X + D_\Delta^* C_\Delta) < 0 \end{aligned}$$

for all $t \geq 0$ and $\Delta \in \boldsymbol{\Delta}$ where $R_\Delta = I - D_\Delta^* D_\Delta > 0$.

Robust performance of interval systems

We shall consider strongly robust $\mathcal{H}_\infty$ performance for a special class of uncertain systems:

$$\begin{aligned} \dot{x} &= A_\Delta x + B_\Delta w \\ z &= C_\Delta x + D_{11} w \end{aligned}$$

where $A_\Delta = A + B_a \Delta C_a$, $B_\Delta = B + B_b \Delta C_b$, and $C_\Delta = C + B_c \Delta C_c$. For simplicity, we shall also assume that the uncertainty matrix $\Delta \in \mathbf{\Delta}$ is real time varying and

$$\mathbf{\Delta} = \left\{ \text{block diag}[\delta_1(t) I_{k_1}, \dots, \delta_m(t) I_{k_m}] : \ \delta_i(t) \in [\underline{\delta}_i, \bar{\delta}_i] \right\}$$

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In the following, we shall denote the vertex set of $\boldsymbol{\Delta}$ as

$$\boldsymbol{\Delta}_{\text{vex}} = \left\{ \text{block diag}[\delta_1 I_{k_1}, \ldots, \delta_m I_{k_m}] : \delta_i = \underline{\delta}_i \ \text{ or } \ \delta_i = \bar{\delta}_i \right\}$$

It is easy to see that there are 2^m vertices in $\boldsymbol{\Delta}_{\text{vex}}$.

A vertex result

Theorem

Consider again the uncertain system. Define

$$R_1 := I - D_{11}^* D_{11}$$

Then the following statements are equivalent:

1. *The system satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion.*

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Theorem

Consider again the uncertain system. Define

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Then the following statements are equivalent:

1. *The system satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion.*
2. $R_1 > 0$ *and there exists an* $X = X^* > 0$ *s.t.:*

$$A_\Delta^* X + X A_\Delta + (X B_\Delta + C_\Delta^* D_{11}) R_1^{-1} (B_\Delta^* X + D_{11}^* C_\Delta) \\ + C_\Delta^* C_\Delta < 0$$

for all $\Delta \in \mathbf{\Delta}_{\mathrm{vex}}$.

A vertex result

Theorem

Consider again the uncertain system. Define

$$R_1 := I - D_{11}^* D_{11}$$

Then the following statements are equivalent:

3. $R_1 > 0$ *and there exists an* $X = X^* > 0$ *s.t.:*

$$\begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_\Delta & B_\Delta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A_\Delta^* & 0 \\ B_\Delta^* & 0 \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} C_\Delta^* C_\Delta & C_\Delta^* D_{11} \\ D_{11}^* C_\Delta & -R_1 \end{pmatrix} < 0$$

for all $\Delta \in \boldsymbol{\Delta}_{vex}$.

Robust state feedback control

Now, we shall consider state feedback controller design such that the closed loop system satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion. For technical reasons, we shall only consider the following class of uncertain systems:

$$\begin{array}{rcl} \dot{x} & = & A_\Delta x + B_\Delta w + B_{2\Delta} u, \quad \Delta \in \boldsymbol{\Delta} \\ z & = & C_\Delta x + D_{11} w + D_{2\Delta} u \\ y & = & x \end{array}$$

where $A_\Delta, B_\Delta, B_{2\Delta}, C_\Delta$, and $D_{2\Delta}$ are any affine matrix functions of Δ as assumed previously.

Robust state feedback control

Theorem (Zhou,Khargonekar,Stoustrup,Niemann)

There exists a state feedback controller such that the closed loop system satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion if and only if $R_1 := I - D_{11}^* D_{11} > 0$ and there exist matrices W and $Y = Y^* > 0$ such that

$$\begin{pmatrix} F_{11}(W,Y) & F_{12}(W,Y) & F_{13}(W,Y) \\ F_{12}(W,Y)^* & -R_1 & 0 \\ F_{13}(W,Y)^* & 0 & -I \end{pmatrix} < 0$$

for all $\Delta \in \boldsymbol{\Delta}_{\mathrm{vex}}$, where:

$$\begin{aligned} F_{11}(W,Y) &= YA_\Delta^* + A_\Delta Y + W^* B_{2\Delta}^* + B_{2\Delta} W \\ F_{12}(W,Y) &= B_\Delta + YC_\Delta^* D_{11} + W^* D_{2\Delta}^* D_{11} \\ F_{13}(W,Y) &= YC_\Delta^* + W^* D_{2\Delta}^* \end{aligned}$$

Furthermore, in that case, one possible choice for F is:

$$F = WY^{-1}.$$