8th semester

Optimal Control

Note on integral action with disturbance- and reference models

February 21, 2013
Feed-forward from measured or feedback using estimated disturbances

Disturbance rejection by the use of feed-forward from measured disturbances is an appealing addition to linear quadratic control which combines nicely with the feedback from measured state variables. The control law is often calculated as feed-forward from an autonomous disturbance model or a model driven by a stochastic disturbance. If no other model is known an integrator with no input or with white noise as input is used. The concept has some inherent shortcomings and will seldom work satisfactory without being combined with other additions which give integral action.

Obviously the disturbance is not controllable since the disturbance state can not be affected from control input. If an integrating model is used to describe the disturbance, standard software (Matlab command lqr) will fail to find a controller since the disturbance state is not stabilisable. This is most of all a formal problem because we do not aim to control the disturbance state, but to calculate a feed-forward which eliminates the effect of the disturbance on the states weighted in the performance.

It is possible to work around the problem by using a disturbance model where integrators are modified by moving the poles from zero slightly inside the unit circle in discrete time or slightly into the left half plane in continuous time. If the controller is calculated by integrating or in discrete time iterating the Riccati equations the above problem appears in the way that the Riccati matrix $S$ does not converge to a constant value but rather tends to increase linearly with integration time, $T$ (or the finite horizon $N$ in the performance sum in discrete time). However the controller gains $L$ may converge to constant values and a stop criterion for the iteration based on this can be used.

However even when a solution is found this does not ensure zero steady state error, but rather the minimization of the quadratic performance with a long horizon will result in a controller which minimize the steady state value of

$$x_{\infty}^TQ_1x_{\infty} + u_{\infty}^TQ_2u_{\infty}$$

In other words the usual performance function does not have a finite value, but the solution may be interpreted as the solution to a problem with the performance function changed to

$$J = \lim_{N \to \infty} E\left\{ \frac{1}{N} \sum_{k=0}^{N} (x(k)^TQ_1x(k) + u(k)^TQ_2u(k)) \right\}$$

or in continuous time

$$J = \lim_{T \to \infty} E\left\{ \frac{1}{T} \int_0^T x(t)^TQ_1x(t) + u(t)^TQ_2u(t)\,dt \right\}$$

**Interpretation of LQG as a two input, two output problem**

Often specification to controller performance include a demand of zero steady state error when the plant is disturbed by persistent signals. These persistent disturbances may be modelled using integrators or other models with continuous time poles on the
imaginary axis or discrete time poles on the unit circle. The general remedy to obtain this goal is to include the disturbance model in the controller using the internal model principle. The internal model principle may be implemented in several ways. Next we will introduce interpretation of the optimal control as a two input two output problem and use this description to illustrate similarities and differences in various formulations of control problems.

The two input two output formulation has a right on its own in the interpretation of optimal control as minimizing a norm of a certain transfer function (matrix) and is similar to the viewpoint used in $H_2$ and $H_\infty$ control. In figure 1 which illustrates this $w = [w_n^T w_d^T]^T$ form the first of the two vectors of inputs, external inputs which can not be controlled. The vector of controllable signals $u$ is the second input. $z = [z_u^T z_x^T]^T$ is the first output which is the (performance) output to be kept small in some sense. The second output $y$ represents everything which can be measured.

A large class of problems may be described in this way of course including control problems but also filtering and detection problems.

We will first consider the LQG problem with stochastic disturbances and stochastic measurement noise. Recognize that if $w_d$ is a stochastic signal with mean zero, $Ew_d = 0$ and covariance $Ew_d(k)w_d^T(k-l) = I\delta(l)$ we obtain

$$R_{ex} = E\{e_x(k)e_x^T(k)\} = E\{\Gamma_dw_d(k)(\Gamma_dw_d(k))^T\} = \Gamma_d\Gamma_d^T$$  \hspace{1cm} (4)

Likewise if $w_n$ is stochastic noise with zero mean and covariance $Ew_n(k)w_n^T(k-l) = I\delta(l)$

$$R_{ey} = E\{e_y(k)e_y^T(k)\} = E\{J_{yn}w_n(k)(J_{yn}w_n(k))^T\} = J_{yn}J_{yn}^T$$  \hspace{1cm} (5)

If we choose

$$Q_1 = H_{xx}^T H_{xx}, \quad Q_2 = J_{zu}^T J_{zu}$$  \hspace{1cm} (6)

we obtain

$$z^T z = \begin{bmatrix} z_u^T & z_x^T \end{bmatrix} \begin{bmatrix} z_u & z_x \end{bmatrix} = x^T Q_1 x + u^T Q_2 u$$  \hspace{1cm} (7)

Figure 1: Interpretation of LQG problem as two input two output problem
So we see that with the choices indicated above design of an LQG controller is equivalent to design of a controller which measures $y$ and gives $u$ as control signal and which minimizes

$$J = \lim_{N \to \infty} E \left\{ \frac{1}{N} \sum_{0}^{N} z(k)^{T} z(k) \right\}$$

when the input to the system is white noise with variance

$$R = E\{w(k)^{T} w(k)\} = E\{\begin{bmatrix} w_n(k)^{T} w_d(k)^{T} \\ w_d(k) \end{bmatrix} \begin{bmatrix} w_n(k) \\ w_d(k) \end{bmatrix}\} = I$$

Because this is equivalent to minimizing the $H_2$ norm of the transfer function matrix from $w$ to $z$ (square root of squared sum of pulse response in discrete time, square root of integral of squared impulse response in continuous time) this is also called $H_2$ control.

The two input two output $H_2$ set-up where the control problem is interpreted as minimization of the norm of a certain transfer function may also be interpreted in a deterministic sense where the inputs $w_d$ and $w_n$ are vectors of unit pulses (or impulses in continuous time). You will also recognize that initial value problems may very easily be defined simply by introducing the initial values of the states using pulses in $e_x$ thus defining sizes of the initial values using $\Gamma_d$

### Direct cancellation of disturbances

It has turned out that the optimal controller does not automatically reject disturbances to give zero steady state error. It is an obvious possibility to search for a more direct approach to find a feed-forward control signals which cancel the disturbances having in mind that this will only cancel the disturbances we can measure and often will need a feedback part to take care of the rest.

We will assume that the number of integrating disturbances is equal to the number of inputs $m$. We will also assume that all states are observable from the output $y$ and that all states can be seen in the performance outputs (all states are also observable through $H_z$). If a steady state exists the entrance to the integrator in the observer will be zero

$$K_d(y_\infty - H_z \hat{x}_\infty) = 0$$

The condition on observability from output imply that $K_d$ has rank $m$ and that the steady state estimate of the integrating disturbance

$$\hat{x}_{d\infty} = x_{d\infty}$$

If we want the steady state value of the weighted output to be zero

$$H_z x_\infty = H_z (\Phi_s x_\infty + \Gamma_d x_{d\infty} + \Gamma_s u_\infty) = 0$$

this can be ensured if we can obtain

$$\Gamma_d x_{d\infty} + \Gamma_s u_\infty = 0$$
If you want to weight as many states through $H_z$ as you have inputs this is also a necessary condition unless there is a zero at 1 in $H_z(zI - \Phi_s)\Gamma_d$. This puts a condition on the control law from integrating disturbance

$$\Gamma_d x_{d\infty} + \Gamma_s (-L_d x_{d\infty}) = 0 \Rightarrow \Gamma_d - \Gamma_s L_d = 0$$

(14)

A simple case is when all integrating disturbances are entering at the input, $\Gamma_d = \Gamma_s \Lambda$ where $\Lambda$ is a diagonal matrix. In that case zero steady state error may be ensured by

$$L_{di} = \Lambda$$

(15)

If we make this choice we also decide on the control law ($L_{di}$) from the integrating disturbance without involving performance minimization using controller Riccati equations. The control law we have obtained might also be found by iterating the Riccati equations with no weight on the control signal, $Q_2 = J_{zu}^T J_{zu} = 0$. This will result in a deadbeat controller and will give the feedback from the ordinary states $x_s$. There are two problems connected to this. First: since the disturbance states are not controllable the use of standard software (Matlab control toolbox lqr) will not provide a solution if the disturbance model is unstable (including integrators). Iteration of the Riccati equations over finite horizon may result in a controller. Alternatively you may move the disturbance poles slightly into the left half plane in the controller design model to obtain the controller. Note that the observer should still be implemented with integrating disturbance model to obtain integral action. Secondly: designing the controller with no weight on the control input force all the controller poles to move to origo in the discrete time case so that the only poles you can influence in your design are those connected to the observer. In the continuous time case there may be no solution. Another somewhat ad hoc choice would be to use value of $L_{di}$ we found above as a precondition and solve the controller Riccati equations in a set-up without integrating disturbance. Note that the value of $L_s$ is independent of the disturbance model.

**Rejection of input or state disturbances as two input two output problem**

In figure 2 the two input two output problem is extended with a model of an integrating disturbance. Remember, that steps or constant disturbances were modelled with this type of model but without input. With a stochastic input, we get integrated white noise, often called random walk. The usual state disturbance without integral action has been combined with the integrated noise since this description has enough degrees of freedom. In the figure an observer based controller has also been added.

Observe now, that this is precisely a problem where we have an uncontrollable disturbance model and a performance function where weighting of $z_u$ in the performance prevents elimination of steady state errors. We will first discuss how we can work around this with the controller structure shown and secondly discuss the design of a controller if we formulate a similar problem which is controllable.

In figure 3 is shown a problem where there is no weight on the control signal $u$. On the other hand we weight changes in the control signal $\Delta u(k) = u(k) - u(k-1)$,
which prevent excessive control signals to reject errors in transients. In the figure
\( \Sigma(z) = (zI - I)^{-1}z \) represents a discrete time integrator and \( \Sigma^{-1}(z) = (I - z^{-1}I) \)
represent discrete time difference. Likewise to simplify the drawing \( \Phi(z) = (zI - \Phi_s)^{-1} \)
represents the plant dynamics.

We can design a controller for this problem by explicitly introducing an integrator for \( \Delta u(k) \) which for the design algorithm is seen as a part of the model such that \( x_u(k) = u(k - 1) \) is introduced as a part of the state vector. This is seen in figure 4.

\[
\begin{bmatrix}
    x_s(k+1) \\
    x_d(k+1) \\
    x_u(k+1)
\end{bmatrix}
= \begin{bmatrix}
    \Phi_s & \Gamma_d & \Gamma_s \\
    0 & I & 0 \\
    0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
    x_s(k) \\
    x_d(k) \\
    x_u(k)
\end{bmatrix}
+ \begin{bmatrix}
    \Gamma_s \\
    0 \\
    I
\end{bmatrix} \Delta u(k) + \begin{bmatrix}
    \Gamma_d \\
    I \\
    0
\end{bmatrix} w_d(k)
\]

The observer should be designed to estimate \( \hat{x}_s \) and \( \hat{x}_d \). Since \( x_u(k) = u(k - 1) \) is
known it need not be included in an observer:

\[
\begin{bmatrix}
    \dot{x}_s(k+1) \\
    \dot{x}_d(k+1)
\end{bmatrix}
= \begin{bmatrix}
    \Phi_s & \Gamma_d \\
    0 & I
\end{bmatrix}
\begin{bmatrix}
    \dot{x}_s(k) \\
    \dot{x}_d(k)
\end{bmatrix}
+ \begin{bmatrix}
    \Gamma_s \\
    0
\end{bmatrix} u(k) + \begin{bmatrix}
    K_s \\
    K_d
\end{bmatrix} \left( y - [H_s, 0] \begin{bmatrix}
    \hat{x}_s(k) \\
    \hat{x}_d(k)
\end{bmatrix} \right)
\]

The calculation of the control signal is done from the equations

\[
\Delta u(k) = [-L_s - L_d - L_u]
\begin{bmatrix}
    \dot{x}_s(k) \\
    \dot{x}_d(k) \\
    u(k-1)
\end{bmatrix}
\]

\[
u(k) = u(k-1) + \Delta u(k)
\]

This set-up allows for design of a controller with reasonable dynamics. The disturbance
is still not controllable, but as indicated earlier this can be solved either by iterating
the Riccati equation over a finite horizon or by modifying the disturbance integrators
by moving the poles slightly inside the unit circle.

Another possibility to set-up a problem which can be solved directly with an infinite
horizon controller is to search for a state space representation which is controllable.
We will discuss two such representations. First we assume that the integrating disturbance
is an input disturbance which give possibility for simple solutions. Secondly we introduce a simple state transformation to obtain controllability.

If the integrating disturbance and the control input span the same subspace a possibility is to combine the disturbance state and controller state in one combined state, for instance \( x_{ad}(k) = \Gamma_d x_d(k) + \Gamma_s x_u(k) \) which will be controllable.

Model :

\[
\begin{bmatrix}
    x_s(k+1) \\
    x_{ad}(k+1)
\end{bmatrix}
= \begin{bmatrix}
    \Phi_s & I \\
    0 & I
\end{bmatrix}
\begin{bmatrix}
    x_s(k) \\
    x_{ad}(k)
\end{bmatrix}
+ \begin{bmatrix}
    \Gamma_s \\
    \Gamma_s
\end{bmatrix} \Delta u(k) + \begin{bmatrix}
    \Gamma_d \\
    \Gamma_d
\end{bmatrix} w_d(k)
\]

Observer:

\[
\begin{bmatrix}
    \dot{x}_s(k+1) \\
    \dot{x}_{ad}(k+1)
\end{bmatrix}
= \begin{bmatrix}
    \Phi_s & I \\
    0 & I
\end{bmatrix}
\begin{bmatrix}
    \dot{x}_s(k) \\
    \dot{x}_{ad}(k)
\end{bmatrix}
+ \begin{bmatrix}
    \Gamma_s \\
    \Gamma_s
\end{bmatrix} \Delta u(k) + \begin{bmatrix}
    K_s \\
    K_{ad}
\end{bmatrix} \left( y - [H_s, 0] \begin{bmatrix}
    \dot{x}_s(k) \\
    \dot{x}_{ad}(k)
\end{bmatrix} \right)
\]

With this model it is straightforward to design both controller and observer, but obtaining such a model is only possible if the integrating disturbance is in the direction of the input: \( \Gamma_d \) and \( \Gamma_s \) span the same subspace of the state space.
If this is not the case it may be more attractive to use \( \Delta x_s(k) = x_s(k) - x_s(k-1) \) as state vector. This implies that we need to use also the outputs of the plant as state variables. We will present a state space model where only \( y \) is taken as output so we assume that the weighted output \( z_x \) is some matrix multiplied by \( y \). Since \( y \) is corrupted by output noise we introduce the state variable \( \eta \) to represent the noise free output.

**Model:**

\[
\begin{bmatrix}
\Delta x_s(k+1) \\
\eta(k+1)
\end{bmatrix}
= \begin{bmatrix}
\Phi_s & 0 \\
H_s \Phi_s & I
\end{bmatrix}
\begin{bmatrix}
\Delta x_s(k) \\
\eta(k)
\end{bmatrix}
+ \begin{bmatrix}
\Gamma_s \\
H_s \Gamma_s
\end{bmatrix}
\Delta u(k)
+ \begin{bmatrix}
\Gamma_d \\
H_s \Gamma_d
\end{bmatrix}
w_d(k)
\]

\[
y(k) = \begin{bmatrix}
0 & I
\end{bmatrix}
\begin{bmatrix}
\Delta x_s(k) \\
\eta(k)
\end{bmatrix}
+ J_{yn} w_n(k)
\]

In this description the integrators from the integrating disturbance have been included in integrators for \( \eta \) and have the possibility to be fully controllable if the number of outputs is less or equal to the number of inputs.

As a last approach as shown in figure 5 integral action is introduced with integrators at the output. On the figure are shown integrators after the 'performance' output \( z_x \), since it is the integrated square of this output plus \( z_{uT}z_u \) we want to minimize. Minimizing the \( H_2 \) norm of the transfer function matrix from \([w_u^T, w_d^T]^T\) to \([z_{uT}, z_{xT}]^T\) is meaningful, but where the problem we had to work around in the formulation from figure 4 was that the integrator after \( w_d \) was not controllable, we now have the dual problem that the integrator after \( z_x \) is not observable. The solution is to use the integral of measured outputs \( y_i \) which we can make observable by implementing the integrator as a part of the controller.
Figure 2: LQG problem with input disturbance as two input two output problem including observer based controller
Figure 3: LQG problem with input disturbance as two input two output problem

Figure 4: LQG problem with input disturbance as two input two output problem

Figure 5: LQG problem with integrated output as two input two output problem